Lecture 24 (3/7/2022)

Important Corollaries to Runge's Thm.

Cor 1. Let $G \subseteq \mathbb{C}$ be region, $E \subseteq \mathbb{C}_\infty \setminus G$ s.t. E meets every component of $\mathbb{C}_\infty \setminus G$. Then, $R(E) = \{$rational fcns w/ poles in $E\}$ is dense in $H(G)$.

Cor 2. Let $G \subseteq \mathbb{C}$ be s.t. $\mathbb{C}_\infty \setminus G$ is connected. Then, polynomials are dense in $H(G)$.

Cor 2 follows from Cor 1. For pf of Cor 1, we recall the existence of an exhaustion $G = \bigcup_{n=1}^{\infty} K_n$, $K_n \subset\subset \operatorname{int} K_{n+1}$,

s.t. $\forall K \subset\subset G$ $\exists K_n$ s.t. $K \subset\subset K_n$,

and every component of $\mathbb{C}_\infty \setminus K_n$ contains a component of $\mathbb{C}_\infty \setminus G$.

By Runge, we can then find $R_n \in R(E)$ s.t.

$$\sup_{K_n} |f - R_n| < \left(\tfrac{1}{2}\right)^n.$$

Since K_n is exhaustion, $R_n \to f$ in $H(G)$.

Def. ① If $K \subset\subset \mathbb{C}$, then the <u>polynomial hull</u>, $\hat{K}$, of K is given by

$$\hat{K} = \{ z \in \mathbb{C} : |p(z)| \leq \sup_K |p| \text{ for all polynomials } p \}.$$

- Clearly, $K \subseteq \hat{K}$. Also, $\hat{K}$ is closed by continuity and bounded since every p satisfies $\lim_{z \to \infty} |p(z)| = \infty$. Thus, $\hat{K}$ is compact.
- An important observation (indeed, the main motivation) is that if $\{p_n\}_{n=1}^{\infty}$ is seq. of polynomials s.t. $p_n \to f$ unif. on K, then $\exists\ \hat{f} \in C(\hat{K})$ s.t. $\hat{f}|_K = f$ and $p_n \to \hat{f}$ unif. on $\hat{K}$.

Def. ② $K \subset\subset \mathbb{C}$ is <u>polynomially convex</u> if $K = \hat{K}$.

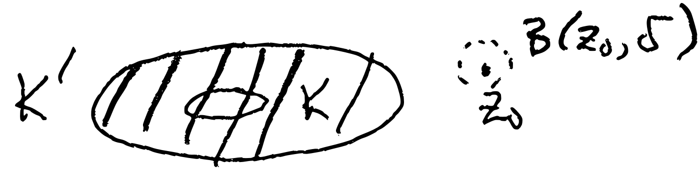
Thm 1. Let U_∞ be unbounded component of $\mathbb{C} \setminus K$. Then $\hat{K} = \mathbb{C} \setminus U_\infty$.

Rem. This means that $\mathbb{C}_\infty \setminus \hat{K}$ is connected (indeed $= U_\infty \cup \{\infty\}$).

Pf. We first show that if U_0 is a bounded component of $\mathbb{C} \setminus K$, then $U_0 \subseteq \hat{K}$. Since K is closed, $\partial U_0 \subseteq K$. By Max Mod Thm, $\sup_{\overline{U_0}} |p| \le \sup_{\partial U_0} |p| \le \sup_K |p|$ $\Rightarrow \overline{U_0} \subseteq \hat{K}$.

Next, we must show that if $z_0 \in U_\infty$, then $z_0 \notin \hat{K}$, i.e. we must find polynomial p s.t. $|p(z_0)| > \sup_K |p|$.

Let $K' = \mathbb{C} \setminus U_\infty \subset\subset \mathbb{C}$ and $K'' = K' \cup \{z_0\} \subset\subset \mathbb{C}$.



Let $\delta = d(z_0, K') > 0$ and consider $G = G' \cup B(z_0, \delta/2)$, $G' = \{z : d(z, K') < \delta/2\}$.

Note that $G' \cap B(z_0, \delta/2) = \emptyset$ and hence,

$$f(z) = \begin{cases} 0, & z \in G' \\ 1, & z \in B(z_0, \delta/2) \end{cases}$$

is in $H(G)$. Since $K'' \subset\subset G$, $\mathbb{C} \setminus K'' = U_\infty \setminus \{z_0\}$, which is connected. Cor 2 of Runge $\Rightarrow$ $\exists$ polynomial p s.t. $\sup_{K''} |f-p| < \varepsilon$ for some small ε ($< 1/4$). But then, $\sup_{K'} |p| < \varepsilon$ and $|p(z_0)| > 1-\varepsilon$

$\Rightarrow z_0 \notin (K')^\wedge \supseteq \hat{K} \Rightarrow \hat{K} \subseteq K'$.

But by first part $K' \subseteq \hat{K} \Rightarrow$

$K' = \hat{K}$ as desired. $\boxtimes$

Equivalent conditions for simple connectedness.

See Thm 2.2. in Ch. VIII of Conway (or Thm in Lecture 23 notes).

We note here those that are not expressed in terms of complex analysis.

Thm. $G \subseteq \mathbb{C}$ region. TFAE:

(i) G is simply connected.

(ii) $\mathbb{C}_\infty \setminus G$ is connected.

(iii) G is homeomorphic to $\mathbb{D} = \{|z|<1\}$.

Pf. (iii) $\Rightarrow$ (i) is clear. (i) $\Rightarrow$ (iii) by Riemann Mapping Thm + fact that $\mathbb{C}$ is homeomorphic to $\mathbb{D}$. (Consider $z \to z/1+|z|$ as homeo $\mathbb{C} \xrightarrow{\cong} \mathbb{D}$.)

To show (i) $\Rightarrow$ (ii), we note that (i) $\Rightarrow$ (by Cauchy's Thm) that $n(z,\gamma)=0$ for all closed curves γ in G, and $z \in \mathbb{C} \setminus G$.

Suppose $\mathbb{C}_\infty \setminus G$ is not connected. Then there are closed, disjoint subsets $(A \neq \emptyset, B \neq \emptyset)$ of $\mathbb{C}_\infty$ s.t. $\mathbb{C}_\infty - G = A \cup B$ and $\infty \in A$. Then, $B \subset\subset \mathbb{C}$ since B closed in $\mathbb{C}_\infty$ and $\infty \notin B$. Thus, $G' = G \cup B = \mathbb{C}_\infty \setminus A$ is open in $\mathbb{C}$. (since $\infty \notin G'$), and $B \subset\subset G'$. By previous result, (in pf of Runge) $\exists$ line segments $\underbrace{\gamma_1, \ldots, \gamma_n}_{\subset\subset G' \setminus B}$ s.t. $\forall z \in B$, $f \in H(G')$,

$$f(z) = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(\zeta)\,d\zeta}{\zeta - z}.$$

Moreover, it is easy to see (DIY) that the γ_j must form finite number of closed polygons $\Gamma_1, \ldots \Gamma_m$ in $G' \setminus B = G$. Since $f \equiv 1$ is in $H(G')$ $\Rightarrow$

$$1 = \sum_{j=1}^{m} n(z, \Gamma_j), \quad z \in B.$$

This contradicts the fact that $n(z,\gamma)=0$ for all $z \in B \subseteq \mathbb{C} \setminus G$ and closed curves $\gamma \in G$. Thus, (i) $\Rightarrow$ (ii).

For (ii) $\Rightarrow$ (iii), we observe that (ii) $\Rightarrow$ $\int_\gamma f\,dz = 0$, for all $f \in H(G)$ and all closed curves γ in G. This follows from homology version of Cauchy's Thm (show $\gamma \approx 0$ in G when $\mathbb{C}_\infty \setminus G$ is connected) or by homotopy version and Cor 2 above (approximate f unif. on γ by polynomials and use $\int_\gamma p\,dz = 0$).

This $\Rightarrow$ every $f \in H(G)$ has a primitive $\Rightarrow$ if $f \neq 0$ in G, then f'/f has a primitive F. A well chosen constant

C, then yields a branch of $\log f$ by $F(z)+C$ $\left(\frac{d}{dz} \frac{e^{F(z)+C}}{f(z)} = 0 \right)$.

Then, we can also define a branch of $\sqrt{f} = e^{\frac{1}{2}\log f}$. We recall that existence of a branch of $\sqrt{f}$, for every $f \in H(G)$ w/ $f \neq 0$, was enough to prove that if $G \neq \mathbb{C}$ then $\exists$ conformal map $G \xrightarrow[\cong]{} \mathbb{D}$. If $G = \mathbb{C}$ then as above $z \to z/_{1+|z|^2}$ is a homeomorphism $\mathbb{C} \to \mathbb{D}$. Thus, (ii) $\Rightarrow$ (iii) and pf is complete. $\blacksquare$